Topology and its Applications 159 (2012) 721–732

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

An explicit L_∞ structure for the components of mapping spaces

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ARTICLE INFO

Article history:
Received 26 April 2011
Received in revised form 3 November 2011
Accepted 4 November 2011

MSC:
primary 55P62
secondary 54C35

Keywords:
Mapping space
L_∞-algebra
Sullivan model
Quillen model
Rational homotopy theory

ABSTRACT

In this paper, we describe an explicit L_∞-algebra structure on the differential graded vector space of Lie γ-derivations between two differential graded Lie algebras where the source is minimal free. If γ is a Lie model for a based map between a finite simply connected complex and a nilpotent complex of finite type over $\mathbb{Q}$, then the above L_∞-algebra is a model for the corresponding component of the space of based maps.

1. Introduction

Rational homotopy type of mapping spaces has been extensively studied since the work of Haefliger [12], where an algebraic model for the space of sections of a nilpotent bundle homotopic to a given one was described. Subsequently, Bousfield, Peterson and Smith constructed in [1] a rational model for the mapping space in a functorial way, via a division functor in the category of commutative differential $\mathbb{Z}$-graded algebras over the rationals. Brown and Szczarba described in [3] this rational model for mapping spaces by computing the division functor explicitly.

All spaces considered in this paper will be nilpotent, so that they admit classical localization. Let $f : X \to Y$ be a based map between a finite simply connected complex X and a complex of finite type Y. We denote by $\mathrm{map}_f(X, Y)$ the path component of the space of continuous maps from X to Y containing f, and by $\mathrm{map}^*_f(X, Y)$ the component in the space of based functions.

The Brown–Szczarba approach has been used in order to obtain, for example, a complete description of the homotopy Lie algebra structure of any component of the mapping space [5]. The description is based on the homology of a complex of derivations between commutative differential graded rational algebras (CDGA from now on) modeling the source X and the target Y. Moreover, this construction allows us to give conditions on X and Y for which all the components $\mathrm{map}^*_f(X, Y)$ are (rationally) H-spaces.

In some cases this complex of derivations is a differential graded rational Lie algebra (DGL from now on) modeling the components of the mapping space [6], and an L_∞-algebra in the general case (see [8] for details).

Recall that the notion of L_∞-algebra, first defined in [21], and the geometrical translation of algebraic properties of these structures, have been successfully applied in several contexts (see, for instance, [9,11,13]).

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[1] The author was supported by the *Ministerio de Ciencia e Innovación* grant MTM2010-15831 and by the *Junta de Andalucía* grant FQM-213.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2011.11.030

In [17,18], rational homotopy groups and the Whitehead product for components of the mapping space were described by means of a complex of derivations between DGL models of X and Y. However, in the spirit of Eckmann–Hilton duality, this complex of Lie derivations has a richer structure. It is a DGL model in some cases, and an L_∞-model for the components of the space of based maps in general [6,7].

More precisely, let $\gamma : L \to L'$ be a DGL morphism and consider the differential graded vector space of Lie γ-derivations $(Der_\gamma(L, L'), \delta)$. Explicitly, $Der_\gamma(L, L')_n$ is the space of linear maps $\theta : L_* \to L'_{*+n}$, such that $\theta[x, y] = [\theta(x), \gamma(y)] + (-1)^{n|x|}[\gamma(x), \theta(y)]$, for every $x, y \in L$. The differential is defined as usual $\delta\theta = \partial \circ \theta + (-1)^{n+1}\theta \circ \partial$.

Consider the complex $\mathcal{D}er_\gamma(L, L')$ of positive γ-derivations,

$$\mathcal{D}er_\gamma\big(L, L'\big)_i = \begin{cases} Der_\gamma(L, L')_i & \text{for } i > 1, \\ ZDer_\gamma(L, L')_1 & \text{for } i = 1, \end{cases}$$

where Z denotes the space of cycles.

Now, choose a Quillen model for $f : X \to Y$ of the form $\gamma : \mathcal{L}(C) \to L'$, where $C = \mathbb{Q} \oplus C_+$ is a coaugmented cocommutative differential graded coalgebra (CDGC from now on) modeling X, and $\mathcal{L}$ is the Quillen functor (see Section 2 for details). We write s for the *suspension* operator as usual and accordingly s^{-1} for the *desuspension* operator. Thus, $(sL)_k = L_{k-1}$. Then the DGL model for $\mathrm{map}^*_f(X, Y)$ given in [6], can be described in terms of derivations [7, Theorem 3]:

Theorem 1.1. *The differential graded Lie algebra $s^{-1}\mathcal{D}er_\gamma(\mathcal{L}(C), L')$ equipped with the usual differential and the bracket given by*

$$[\theta, \eta](c) = \sum_i (-1)^{1+|a_i||\eta|}\big[\theta(a_i), \eta(b_i)\big]$$

is a Lie model for $\mathrm{map}^*_f(X, Y)$, where $\overline{\Delta}(c) = \sum_i a_i \otimes b_i$ and $\overline{\Delta}$ denotes the reduced diagonal.

If $\gamma : L \to L'$ a Quillen model of $f : X \to Y$, then the following result is proved in [7, Theorem 5]:

Theorem 1.2. *There is an L_∞ structure on $s^{-1}\mathcal{D}er_\gamma(L, L')$ for which it becomes an L_∞-model for $\mathrm{map}^*_f(X, Y)$.*

However, in the previous theorem, the existence of such an L_∞ structure is deduced from a deformation of the DGL structure of $s^{-1}\mathcal{D}er_\gamma(\mathcal{L}(C), L')$ via the *homotopy transfer theorem* [16, §10.3], and higher order brackets are deeply hidden.

In this paper, we give an explicit description of this structure in terms of the DGL structure of the data. In fact, when $\mathbb{L}(\mathcal{U})$ is the minimal Lie model of X and with the operations introduced in Definition 3.1 (which are defined by duality following the operations given in [5]), we prove the following:

Theorem 1.3. *The complex $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}), L')$ is an L_∞-algebra whose geometrical realization is $\mathrm{map}^*_f(X, Y)$.*

At first sight, the obvious approach to prove this theorem would be using the explicit formula for the transferred L_∞-algebra structure induced by a homotopy retract [16, Theorem 10.3.9]. But it is not clear which homotopy retract data will produce the announced L_∞ structure.

Our explicit structure can be applied to bound the Whitehead-length and to detect H-space structures in the components of the mapping space.

We briefly recall the definition of the following invariants. The *rational bracket-length* of X, denoted by $\mathrm{bl}_{\mathbb{Q}}(X)$, is the length of the shortest non-zero bracket in the differential of the Quillen minimal model $(\mathbb{L}(\mathcal{U}), \partial)$ of X. If the differential is zero, we define $\mathrm{bl}_{\mathbb{Q}}(X) = \infty$. The *rational Whitehead-length* of a space Z, denoted $\mathrm{Wh}_{\mathbb{Q}}(Z)$, is the length of the longest, non-zero iterated Whitehead bracket in $\pi_{\geqslant 2}(Z) \otimes \mathbb{Q}$. In particular, $\mathrm{Wh}_{\mathbb{Q}}(Z) = 1$ means that all Whitehead products vanish.

Recall that a nilpotent topological space Y of finite type is called *coformal* if its rational homotopy Lie algebra $(\pi_*(\Omega Y) \otimes \mathbb{Q}, 0)$ is a DGL model for it. Then we prove the following result:

Theorem 1.4. *Let Y be a coformal space. Then*:

(1) *For all $f : X \to Y$ we have that*

$$\mathrm{Wh}_{\mathbb{Q}}\big(\mathrm{map}^*_f(X, Y)\big) \leqslant \frac{\mathrm{Wh}_{\mathbb{Q}}(Y) - 1}{\mathrm{bl}_{\mathbb{Q}}(X) - 1} + 1.$$

(2) *If $\mathrm{bl}_{\mathbb{Q}}(X) > \mathrm{Wh}_{\mathbb{Q}}(Y)$, then $\mathrm{map}^*_f(X, Y)_{\mathbb{Q}}$ is an H-space for all $f : X \to Y$. Equivalently, its rational cohomology algebra is free.*

The paper is organized as follows: Section 2 is a compendium of known results about rational homotopy theory, L_∞-algebras and Sullivan models for mapping spaces. Theorem 1.3 is proved in Section 3, where we show that the commutative differential graded algebra $\mathcal{C}^\infty(s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}), L'))$ is isomorphic to the Brown–Szczarba model for the corresponding component (see Section 2 for definitions and further details). Finally, Section 4 is devoted to prove Theorem 1.4 and to give some geometrical examples.

2. Background

2.1. Rational homotopy theory

We will heavily rely on known results from rational homotopy theory. For that [10] is a standard and excellent reference. Here, mainly to set the framework in which we work and to fix notation, we recall some basics.

An equivalence between the rational homotopy category of nilpotent spaces with rational homology of finite type, and the homotopy category of CDGA of finite type has been proved by Sullivan [22] and Bousfield and Guggenheim [2]. Earlier, Quillen [20] did the same for the homotopy categories of simply connected spaces, cocommutative differential graded simply connected rational coalgebras, and differential graded connected rational Lie algebras. Later, Neisendorfer [19] combined these two approaches and generalized the above portion of Quillen's work to nilpotent spaces of finite type.

To each nilpotent space X of finite type we can associate three types of differential graded models: a commutative algebra B, a cocommutative coalgebra C, and a free Lie algebra $\mathcal{L}(C)$. When X is simply connected, we can replace $\mathcal{L}(C)$ by a minimal Lie algebra $\mathbb{L}(\mathcal{U})$.

The differentials in the above settings shall be denoted by d, δ and ∂ for algebras, coalgebras and Lie algebras, respectively.

The *Quillen functor* $\mathcal{L}$ is constructed for any coaugmented cocommutative differential graded coalgebra. If (C,δ) is any CDGC, then $\mathcal{L}(C,\delta) = (\mathbb{L}(s^{-1}C_+), \partial)$. The differential ∂ is defined as $\partial_1 + \partial_2$, with $\partial_1(s^{-1}c) = -s^{-1}\delta c$ and

$$\partial_2\big(s^{-1}c\big) = \frac{1}{2}\sum_i (-1)^{|a_i|}\big[s^{-1}a_i, s^{-1}b_i\big],$$

where $\overline{\Delta}c = \sum_i a_i \otimes b_i$, and $\overline{\Delta}: C_+ \to C_+ \otimes C_+$ is the reduced diagonal. From now on we shall write, for convenience, $\mathcal{L}(C)$ instead of $\mathcal{L}(C,\delta)$.

A *free model* of a connected chain Lie algebra $(L = \{L_i\}_{i\geqslant 1}, \partial)$ is a DGL quasi-isomorphism of the form $m: (\mathbb{L}(\mathcal{U}), \partial) \to (L, \partial)$ with $\mathcal{U} = \{\mathcal{U}_i\}_{i\geqslant 1}$. A free model $(\mathbb{L}(\mathcal{U}), \partial)$ is called *minimal* if the linear part ∂_1 of the differential is zero.

Theorem 2.1. *Every connected chain Lie algebra (L, ∂) admits a minimal free Lie model which is unique up to isomorphism.*

For a complete proof of this result we refer to [10, Theorem 22.13]. Here we will give a procedure to obtain a minimal model from a DGL of the form $(\mathcal{L}(B^\sharp), \partial = \partial_1 + \partial_2)$, where B is a finite-dimensional CDGA and $B^\sharp$ denotes the dual CDGC.

First, we decompose $B^\sharp_+$ as $A \oplus \delta A \oplus H_+$, with $\delta = 0$ in H_+ and $\delta: A \xrightarrow{\cong} \delta A$. Then, $H_+ \cong H(B^\sharp_+)$. Now, let $I \subset \mathcal{L}(B^\sharp)$ be the differential ideal generated by $s^{-1}A$ and $\partial s^{-1}A$. Thus, I is preserved by ∂ and the quotient $(\mathbb{L}(s^{-1}B^\sharp_+)/I, \bar{\partial})$ is a connected DGL. It is clear that $\mathbb{L}(s^{-1}B^\sharp_+) = \mathbb{L}(s^{-1}H_+) \oplus I$ and hence the quotient map

$$\pi: \big(\mathbb{L}\big(s^{-1}B^\sharp_+\big), \partial\big) \to \big(\mathbb{L}\big(s^{-1}B^\sharp_+\big)/I, \bar{\partial}\big)$$

restricts to an isomorphism $\varphi: \mathbb{L}(s^{-1}H_+) \xrightarrow{\cong} \mathbb{L}(s^{-1}B^\sharp_+)/I$. We use this isomorphism to identify $(\mathbb{L}(s^{-1}B^\sharp_+)/I, \bar{\partial})$ as a free DGL, namely $(\mathbb{L}(s^{-1}H_+), \partial)$. Then one can check that $(\mathbb{L}(s^{-1}H_+), \partial)$ is minimal, and that the composition $q = \varphi^{-1} \circ \pi$ is a quasi-isomorphism.

Finally, lift $\mathrm{Id}_{(\mathbb{L}(s^{-1}H_+),\partial)}$ through q [10, Prop. 22.12] to obtain a quasi-isomorphism $j: (\mathbb{L}(\mathcal{U}), \partial) \to (\mathcal{L}(B^\sharp), \partial)$, where $\mathcal{U} \cong s^{-1}H_+$. This is in fact the minimal Lie model. In order to simplify notation, we will write $\lambda(z)$ to denote $\varphi^{-1}(\overline{s^{-1}z})$. Then the differential in the minimal Lie model is defined as

$$\partial s^{-1}h = \varphi^{-1}\bar{\partial}\varphi s^{-1}h = \frac{1}{2}\sum_j (-1)^{|z'_j|}\big[\lambda\big(z'_j\big), \lambda\big(z''_j\big)\big],$$

where $\overline{\Delta}h = \sum_j z'_j \otimes z''_j$. The morphism φ^{-1} can be explicitly described as follows. If $h \in H_+$, then $\lambda(h) = s^{-1}h$, and if $a \in A$, then

$$\lambda(\delta a) = \frac{1}{2}\sum_j (-1)^{|z'_j|}\big[\lambda\big(z'_j\big), \lambda\big(z''_j\big)\big], \tag{2.1}$$

where $\overline{\Delta}a = \sum_j z'_j \otimes z''_j$.

2.2. L_∞-algebras

For a basic compendium of known properties of L_∞-algebras, we refer to [14] or [15]. Also, in [13] the algebraic behavior of these structures is nicely introduced as a result of their geometrical counterparts. We follow the notation of [7].

Definition 2.2. An *L_∞-algebra* or *strongly homotopy Lie algebra* is a graded vector space L together with a differential graded coalgebra structure on ΛsL, the cofree graded cocommutative coalgebra generated by the suspension. The existence of a CDGC structure on ΛsL is equivalent to the existence of degree $k-2$ linear maps $\ell_k : L^{\otimes k} \to L$, for $k \geqslant 1$, satisfying the following two conditions:

(1) For any permutation σ of k elements,

$$\ell_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \epsilon_\sigma \epsilon' \ell(x_1, \ldots, x_k),$$

where ϵ_σ is the signature of the permutation and ϵ' is the sign given by the Koszul convention.

(2) The *generalized Jacobi identity* holds, that is

$$\sum_{i+j=n+1} \sum_{\sigma \in S(i,n-i)} \varepsilon \ell_{n-i}\big(\ell_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}\big) = 0,$$

where $\varepsilon = \epsilon_\sigma \epsilon' (-1)^{i(j-1)}$, and $S(i, n-1)$ denotes the set of $(i, n-i)$-shuffles, i.e., permutations σ of n-elements such that $\sigma(1) < \cdots < \sigma(i)$ and $\sigma(i+1) < \cdots < \sigma(n)$.

Every differential graded Lie algebra (L, ∂) is an L_∞-algebra by setting $\ell_1 = \partial$, $\ell_2 = [-,-]$ and $\ell_k = 0$ for $k > 2$.

Observe that if L is a finite type graded vector space, then an L_∞ structure on L is equivalent to a CDGA structure on $(\Lambda sL)^\sharp \cong \Lambda(sL)^\sharp$. We will denote this CDGA structure by $\mathcal{C}^\infty(L)$. This can be seen as a generalization of the cochain algebra on a DGL.

If V and W are dual vector spaces, then the pairing $\langle ; \rangle : V \times W \to \mathbb{Q}$ defined as $\langle v; w\rangle = v(w)$, can be extended to $(k+1)$-linear maps $\Lambda^k V \times W \times \cdots \times W \to \mathbb{Q}$,

$$\langle v_1 \wedge \cdots \wedge v_k; w_k, \ldots, w_1\rangle = \sum_{\sigma \in S_k} \epsilon_\sigma \langle v_{\sigma(1)}; w_1\rangle \cdots \langle v_{\sigma(k)}; w_k\rangle.$$

Then, the following holds:

Proposition 2.3. *Let L be an L_∞-algebra of finite type. Then, $\mathcal{C}^\infty(L) = (\Lambda V, d)$ is a CDGA, where V and sL are dual graded vector spaces and $d = \sum_{j \geqslant 1} d_j$ satisfies $\langle d_j v; sx_1 \wedge \cdots \wedge sx_j\rangle = \epsilon\langle v; s[x_1, \ldots, x_j]\rangle$, with ϵ the sign provided by the Koszul convention. Conversely, suppose $(\Lambda V, d)$ is an arbitrary CDGA of finite type. Then, an L_∞-algebra structure in $s^{-1}V^\sharp$ is uniquely determined by the condition $(\Lambda V, d) = \mathcal{C}^\infty(L)$.*

Moreover, in the above correspondence, $(\Lambda V, d)$ is a Sullivan algebra if and only if L is concentrated in non-negative degrees and L_0 acts nilpotently in L via ℓ_2.

Remark. If L is a connected DGL of finite type, then $\mathcal{C}^\infty(L) = (\Lambda V, d)$, where $d = d_1 + d_2$, is the classical cochain functor [10, §23]. In this case $\langle d_1 v; sx\rangle = (-1)^{|v|}\langle v; s\partial x\rangle$ and $\langle d_2 v; sx \wedge sy\rangle = (-1)^{|y|+1}\langle v; s[x,y]\rangle$.

Definition 2.4. Given a finite type complex X, an L_∞-algebra L is an *L_∞-model* of X if $\mathcal{C}^\infty(L)$ is a Sullivan model of X.

2.3. Models of mapping spaces

We start by introducing the Haefliger model [12] of the mapping space via the functorial description of Brown and Szczarba [3].

Let B be a finite-dimensional CDGA model of the nilpotent finite complex X, and let $(\Lambda V, d)$ be a Sullivan (non-necessarily minimal) model of the nilpotent space Y. We denote by $B^\sharp$ the differential coalgebra dual of B with the grading $B^{\sharp -n} = B^\sharp_n = Hom(B^n, \mathbb{Q})$, and consider $\Lambda(\Lambda V \otimes B^\sharp)$ the free commutative algebra generated by the $\mathbb{Z}$-graded vector space $\Lambda V \otimes B^\sharp$, endowed with the differential d induced by the ones on $(\Lambda V, d)$ and on $B^\sharp$. Let J be the differential ideal generated by $1 \otimes 1 - 1$, and the elements of the form

$$v_1 v_2 \otimes \beta - \sum_j (-1)^{|v_2||\beta'_j|}\big(v_1 \otimes \beta'_j\big)\big(v_2 \otimes \beta''_j\big), \quad v_1, v_2 \in V$$

where $\Delta\beta = \sum_j \beta'_j \otimes \beta''_j$. Then, the map induced by the inclusion $V \otimes B^\sharp \hookrightarrow \Lambda V \otimes B^\sharp$, i.e.

$$\rho : \Lambda\big(V \otimes B^\sharp\big) \xrightarrow{\cong} \Lambda\big(\Lambda V \otimes B^\sharp\big)/J$$

is an isomorphism of graded algebras, and thus $\tilde{d} = \rho^{-1} d \rho$ defines a differential in $\Lambda(V \otimes B^\sharp)$. The following result is proved in [3,12]:

Theorem 2.5. *The commutative differential graded algebra $(\Lambda(V \otimes B^\sharp), \tilde{d})$ is a model of* $\mathrm{map}(X, Y)$*, and the commutative differential graded algebra $(\Lambda(V \otimes B^\sharp_+), \tilde{d})$ is a model of* $\mathrm{map}^*(X, Y)$.

Similarly as we explained after Theorem 2.1, we can decompose $B^\sharp$ as $A \oplus \delta A \oplus H$, with basis given by $\{a_j\}$, $\{b_j\}$ and $\{h_k\}$, and $H \cong H(B^\sharp)$. Thus $\delta a_j = b_j$ and $\delta h_k = 0$. Additionally, since $(\Lambda V, d)$ is a Sullivan algebra, we can choose a basis $\{v_i\}$ for V where $dv_i \in \Lambda V_{<i}$. Then the following holds [3, Lemma 5.1]:

Lemma 2.6. *The commutative differential graded algebra $(\Lambda(V \otimes B^\sharp), \tilde{d})$ splits as*

$$\big(\Lambda\big(V \otimes B^\sharp\big), \tilde{d}\big) = (\Lambda W, \tilde{d}) \otimes \Lambda(U \oplus \tilde{d}U),$$

where:

(1) *U is generated by $u_{ij} = v_i \otimes a_j$,*
(2) *W is generated by $w_{ik} = v_i \otimes h_k - x_{ik}$, for suitable $x_{ik} \in \Lambda(V_{<i} \otimes B^\sharp)$,*
(3) *$\tilde{d} w_{ik} \in \Lambda\{w_{mk}\}_{m<i}$,*
(4) *if $\tilde{d}(v_i \otimes h_k)$ is decomposable, so is $\tilde{d} w_{ik}$.*

Proof. We proceed by induction on i. Suppose that w_{mk} have been defined for $m < i$ satisfying the lemma for $(\Lambda(V_{<i} \otimes B^\sharp), \tilde{d})$. Now, since $\tilde{d}(v_i \otimes h_k) = \rho^{-1}[dv_i \otimes h_k]$ belongs to $\Lambda(V_{<i} \otimes B^\sharp)$, and

$$\Lambda\big(V_{<i} \otimes B^\sharp\big) = \Lambda\{w_{mk}\}_{m<i} \otimes \Lambda\{u_{mj}, \tilde{d}u_{mj}\}_{m<i},$$

we can write $\tilde{d}(v_i \otimes h_k) = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 \in \Lambda\{w_{mk}\}_{m<i} \quad \text{and} \quad \Gamma_2 \in \Lambda\{w_{mk}\}_{m<i} \otimes \Lambda^+\{u_{mj}, \tilde{d}u_{mj}\}_{m<i}.$$

The ideal $\Lambda\{w_{mk}\}_{m<i} \otimes \Lambda^+\{u_{mj}, \tilde{d}u_{mj}\}_{m<i}$ is acyclic, since by inductive hypothesis $\Lambda\{w_{mk}\}_{m<i}$ is $\tilde{d}$-stable. Therefore, Γ_2 is a boundary, i.e., $\Gamma_2 = \tilde{d}x_{ik}$ for some $x_{ik} \in \Lambda\{w_{mk}\}_{m<i} \otimes \Lambda^+\{u_{mj}, \tilde{d}u_{mj}\}_{m<i}$. We define $w_{ik} = v_i \otimes h_k - x_{ik}$, which clearly satisfies (2), (3) and (4). To finish, observe that $\tilde{d}u_{ij} = \tilde{d}(v_i \otimes a_j) = \pm v_i \otimes b_j + \rho^{-1}[(dv_i) \otimes a_j]$, where $\rho^{-1}[(dv_i) \otimes a_j] \in \Lambda(V_{<i} \otimes B^\sharp)$. Hence,

$$\Lambda\big(V_{\leqslant i} \otimes B^\sharp\big) = \Lambda\{w_{mk}\}_{m\leqslant i} \otimes \Lambda\{u_{mj}, \tilde{d}u_{mj}\}_{m\leqslant i}. \qquad \square$$

We can endow the free algebra $\Lambda(V \otimes H)$ with a differential $\hat{d}$ so that the map

$$\sigma : \big(\Lambda(V \otimes H), \hat{d}\big) \xrightarrow{\cong} (\Lambda W, \tilde{d})$$

is an isomorphism of CDGA's, and therefore $(\Lambda(V \otimes H), \hat{d})$ is a model of $\mathrm{map}(X, Y)$.

In the case that $(\Lambda V, d = d_1 + d_2)$ has a differential with only linear and quadratic part, we can easily describe $\hat{d}$ as follows:

Consider the composition $\wp = \sigma^{-1} p : \Lambda(V \otimes B^\sharp) = \Lambda W \otimes \Lambda(U \oplus \tilde{d}U) \to \Lambda(V \otimes H)$, where $p : \Lambda W \otimes \Lambda(U \oplus \tilde{d}U) \xrightarrow{\simeq} \Lambda W$ is the projection.

Lemma 2.7. *The morphism $\wp = \sigma^{-1} p$ operates on the generators as follows*:

$$\wp(v \otimes b) = \begin{cases} v \otimes b & \text{if } b \in H, \\ 0 & \text{if } b \in A, \\ (-1)^{|v|+1} \sum_{i,j} (-1)^{|w_i||z'_j|} \wp(u_i \otimes z'_j) \wp(w_i \otimes z''_j) & \text{if } b \in \delta A, \end{cases}$$

where in the last case, $b = \delta a$, $\Delta a = \sum_j z'_j \otimes z''_j$ and $d_2 v = \sum_i u_i w_i$.

Proof. If $b \in H$, then $\wp(v \otimes b) = \wp(v \otimes b - x_{vb} + x_{vb}) = \wp(v \otimes b - x_{vb}) + \wp(x_{vb}) = v \otimes b$. If $b \in A$, then $v \otimes b \in U$ and trivially $\wp(v \otimes b) = 0$. Assume now $b = \delta a$ for some $a \in A$. Then,

$$v \otimes \delta a = (-1)^{|v|}\tilde{d}(v \otimes a) + (-1)^{|v|+1} d_1 v \otimes a + (-1)^{|v|+1} \sum_{i,j} (-1)^{|w_i||z'_j|} \big(u_i \otimes z'_j\big)\big(w_i \otimes z''_j\big),$$

where $d_2 v = \sum_i u_i w_i$ and $\Delta a = \sum_j z'_j \otimes z''_j$. Since $\tilde{d}(v \otimes a)$ and $d_1 v \otimes a \in \Lambda(U \oplus \tilde{d}U)$, we have that

$$\wp(v \otimes b) = (-1)^{|v|+1} \sum_{i,j} (-1)^{|w_i||z'_j|} \wp\big(u_i \otimes z'_j\big)\wp\big(w_i \otimes z''_j\big). \qquad \square$$

Lemma 2.8. *The differential in the free commutative differential graded algebra $(\Lambda(V \otimes H), \hat{d})$ is given by the formula*

$$\hat{d}(v \otimes h) = d_1 v \otimes h + \sum_{i,j} (-1)^{|w_i||z'_j|} \wp\big(u_i \otimes z'_j\big)\wp\big(w_i \otimes z''_j\big),$$

where, as before, $d_2 v = \sum_i u_i w_i$ and $\Delta h = \sum_j z'_j \otimes z''_j$.

Proof.

$$\begin{aligned}\hat{d}(v \otimes h) &= \sigma^{-1}\tilde{d}(v \otimes h - x_{vh}) = \sigma^{-1} p\tilde{d}(v \otimes h - x_{vh}) = \wp\tilde{d}(v \otimes h - x_{vh}) \\ &= \wp\tilde{d}(v \otimes h) = d_1 v \otimes h + \sum_{i,j} (-1)^{|w_i||z'_j|} \wp\big(u_i \otimes z'_j\big)\wp\big(w_i \otimes z''_j\big),\end{aligned}$$

where we have used that $\tilde{d}x_{vh} \in \Lambda W \otimes \Lambda^+(U \oplus \tilde{d}U)$; see the proof of Lemma 2.6. $\square$

For the model of the components of $\operatorname{map}(X, Y)$ and $\operatorname{map}^*(X, Y)$ we follow the approach and notation of [4]:

For any free CDGA $(\Lambda S, d)$, in which S is $\mathbb{Z}$-graded, and any algebra morphism $u : \Lambda S \to \mathbb{Q}$, consider the differential ideal K_u generated by $A_1 \cup A_2 \cup A_3$, where

$$A_1 = S^{<0}, \qquad A_2 = dS^0, \quad \text{and} \quad A_3 = \big\{\alpha - u(\alpha)\colon \alpha \in S^0\big\}.$$

$(\Lambda S, d)/K_u$ is again a free CDGA. Indeed, let $\tilde{K_u}$ be the ideal of $(\Lambda S, d)$ generated by $A_1 \cup A_3$ and observe that the projection $p : S^1 \xrightarrow{\cong} (\Lambda S/\tilde{K_u})^1$ is an isomorphism of vector spaces. Consider the linear map

$$D : S^0 \xrightarrow{d} (\Lambda S)^1 \twoheadrightarrow (\Lambda S/\tilde{K_u})^1 \xrightarrow{p^{-1}} S^1$$

and call $\bar{S}^1$ a complement of the image of this map, i.e. $S^1 = DS^0 \oplus \bar{S}^1$.

Proposition 2.9. *([4]) There exists a differential d_u for which there is an isomorphism of commutative differential graded algebras $(\Lambda S/K_u, d) \cong (\Lambda(\bar{S}^1 \oplus S^{\geqslant 2}), d_u)$.*

The above isomorphism is induced by $\eta : \Lambda S \to \Lambda(\bar{S}^1 \oplus S^{\geqslant 2})$

$$\eta(w) = \begin{cases} 0 & \text{if } w \in S^{<0} \oplus DS^0, \\ u(w) & \text{if } w \in S^0, \\ w & \text{otherwise.} \end{cases} \tag{2.2}$$

We now apply this constructions to the following situation. Let $f : X \to Y$ be a map modeled by $\phi : (\Lambda V, d) \to (B, d)$, which determines an algebra morphism denoted also by $\phi : (\Lambda(V \otimes B^\sharp), \tilde{d}) \to \mathbb{Q}$. Then:

Theorem 2.10. *([3,4]) The projection*

$$\big(\Lambda\big(V \otimes B^\sharp\big), \tilde{d}\big) \to \big(\Lambda\big(V \otimes B^\sharp\big), \tilde{d}\big)/K_\phi \cong \big(\Lambda\big(\overline{V \otimes B^\sharp}^1 \oplus \big(V \otimes B^\sharp\big)^{\geqslant 2}\big), d_\phi\big)$$

is a model of the inclusion $\operatorname{map}_f(X, Y) \hookrightarrow \operatorname{map}(X, Y)$. Moreover, a model for the inclusion $\operatorname{map}^*_f(X, Y) \hookrightarrow \operatorname{map}^*(X, Y)$ is obtained by the above projection replacing $B^\sharp$ with $B^\sharp_+$.

In the splitting $(\Lambda(V \otimes B^\sharp), \tilde{d}) = (\Lambda W, \tilde{d}) \otimes \Lambda(U \oplus \tilde{d}U)$ we denote by K_{ϕ_W} the differential ideal in $(\Lambda W, \tilde{d})$ associated to

$$\phi_W : (\Lambda W, \tilde{d}) \hookrightarrow (\Lambda W, \tilde{d}) \otimes \Lambda(U \oplus \tilde{d}U) \xrightarrow{\phi} \mathbb{Q};$$

and by K_{ϕ_U} the differential ideal in $\Lambda(U \oplus \tilde{d}U)$ associated to

$$\phi_U : \Lambda(U \oplus \tilde{d}U) \hookrightarrow (\Lambda W, \tilde{d}) \otimes \Lambda(U \oplus \tilde{d}U) \xrightarrow{\phi} \mathbb{Q}.$$

Then, $K_\phi = K_{\phi_W} \otimes K_{\phi_U}$, and therefore

$$\big(\Lambda\big(V \otimes B^\sharp\big), \tilde{d}\big)/K_\phi = (\Lambda W, \tilde{d})/K_{\phi_W} \otimes \Lambda(U \oplus \tilde{d}U)/K_{\phi_U}.$$

However, one easily checks that $\Lambda(U \oplus \tilde{d}U)/K_{\phi_U} \cong \Lambda(U^{\geqslant 1} \oplus \tilde{d}(U^{\geqslant 1}))$ is a contractible CDGA and hence

$$(\Lambda S_\phi, d_\phi) \simeq (\Lambda W, \tilde{d})/K_{\phi_W} \cong \big(\Lambda \overline{W}^1 \oplus W^{\geqslant 2}, d_{\phi_W}\big).$$

Finally, consider the morphism $\psi : (\Lambda(V \otimes H), \hat{d}) \xrightarrow{\sigma} (\Lambda W, \tilde{d}) \xrightarrow{\phi_W} \mathbb{Q}$, which defines the corresponding differential ideal K_ψ. Obviously, σ induces an isomorphism $\overline{\sigma} : (\Lambda(V \otimes H), \hat{d})/K_\psi \to (\Lambda W, \tilde{d})/K_{\phi_W}$ and therefore the isomorphism induced by (2.2)

$$\overline{\eta} : \big(\Lambda(V \otimes H), \hat{d}\big)/K_\psi \xrightarrow{\cong} \big(\Lambda \overline{V \otimes H}^1 \oplus (V \otimes H)^{\geqslant 2}, \hat{d}_\psi\big)$$

provides a Sullivan model of $\mathrm{map}_f(X, Y)$. Exactly the same argument can be applied for the pointed case $\mathrm{map}^*_f(X, Y)$ by replacing H with H_+.

3. L_∞-algebras of derivations modeling mapping spaces

Let $\gamma : L \to L'$ be a DGL morphism and consider the differential graded vector space of positive Lie γ-derivations $(\mathcal{D}er_\gamma(L, L'), \delta)$ with the usual differential $\delta\theta = \partial \circ \theta + (-1)^{n+1}\theta \circ \partial$. (Note that we have made an exception with the notation δ in this context to avoid confusion with the DGL differentials ∂ of L and L'.)

For any free differential graded Lie algebra $L = (\mathbb{L}(V), \partial)$ one can define natural higher brackets $\{\ell_k\}_{k\geqslant 1}$ on the complex $s^{-1}\mathcal{D}er_\gamma(L, L')$. First, given $\theta_1, \ldots, \theta_n$ in $\mathcal{D}er_\gamma(L, L')$, we define, in a recursive way, a linear map of degree $|\theta_1| + \cdots + |\theta_n|$ denoted by

$$\{\theta_1, \ldots, \theta_n\} : \mathbb{L}(V) \longrightarrow M,$$

as follows. If $n = 1$, then $\{\theta_1\} = \theta_1$. For $n \geqslant 2$,

1. if $\Psi \in \mathbb{L}^{<n}(V)$, then $\{\theta_1, \ldots, \theta_n\}(\Psi) = 0$;
2. if $\Psi \in \mathbb{L}^{\geqslant n}(V)$ and $\Psi = [\alpha, \beta]$, where $\alpha, \beta \in \mathbb{L}^{<n}(V)$ then

$$\begin{aligned}
\{\theta_1, \ldots, \theta_n\}(\Psi) &= \{\theta_1, \ldots, \theta_n\}\big([\alpha, \beta]\big) \\
&= \big[\{\theta_1, \ldots, \theta_n\}\alpha, \gamma\beta\big] + (-1)^{|\alpha|(|\theta_1|+\cdots+|\theta_n|)}\big[\gamma\alpha, \{\theta_1, \ldots, \theta_n\}\beta\big] \\
&\quad + \sum_{k=1, \sigma\in S(k,n-k)}^{n-1} \big(\epsilon'\big[\{\theta_{\sigma(1)}, \ldots, \theta_{\sigma(k)}\}\alpha, \{\theta_{\sigma(k+1)}, \ldots, \theta_{\sigma(n)}\}\beta\big] \\
&\quad + \epsilon''\big[\{\theta_{\sigma(k+1)}, \ldots, \theta_{\sigma(n)}\}\alpha, \{\theta_{\sigma(1)}, \ldots, \theta_{\sigma(k)}\}\beta\big]\big),
\end{aligned}$$

where ϵ' and ϵ'' are given by the Koszul sign convention.

Definition 3.1. The brackets $\{\ell_k\}_{k\geqslant 1}$ on $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(V), L')$ are defined in the following way. The first bracket is the differential of the complex $\ell_1(s^{-1}\theta) = -s^{-1}\delta\theta$, and for $n > 1$,

$$s\ell_n\big(s^{-1}\theta_1, \ldots, s^{-1}\theta_n\big)(v) = (-1)^\epsilon\{\theta_1, \ldots, \theta_n\}(\partial v), \tag{3.1}$$

where $\epsilon = n + 1 + \sum_{j=1}^n (n + 1 - j)|\theta_j|$.

If $L = (\mathbb{L}(\mathcal{U}), \partial)$ is the minimal Lie model, then we have the following:

Theorem 3.2. *With the brackets of the above definition $M = s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}), L')$ is an L_∞-algebra of finite type concentrated in non-negative degrees, and M_0 acts nilpotently on M via ℓ_2.*

We will not prove this directly. Instead we will rely on Proposition 2.3 and show that the above brackets are determined uniquely by the differential of a Sullivan algebra of finite type.

More precisely, let B be a 1-connected finite-dimensional CDGA with $\mathbb{L}(\mathcal{U}) = \mathbb{L}(s^{-1}H_+)$ the minimal Lie model of $(\mathcal{L}(B^\sharp), \partial)$. Consider the Sullivan algebra $(\Lambda V, d) = \mathcal{C}^*(L')$ where L' is a nilpotent DGL of finite type, and $\gamma : \mathbb{L}(s^{-1}H_+) \to L'$ a DGL morphism. This morphism can be composed with the quasi-isomorphism $q = \varphi^{-1} \circ \pi$ described after Theorem 2.1,

$$\big(\mathcal{L}\big(B^\sharp\big),\partial\big)\xrightarrow{q}\big(\mathbb{L}\big(s^{-1}H_+\big),\partial\big)\xrightarrow{\gamma}L',$$

which induces the CDGA morphism $\phi:\Lambda(sL')^\sharp\to B_+$. Now, we apply to this morphism the construction described at the end of Section 2:

$$\psi:\Lambda(V\otimes H_+)\xrightarrow{\sigma}\Lambda W\hookrightarrow\Lambda W\otimes\Lambda(U\oplus\tilde{d}U)\cong\Lambda\big(V\otimes B_+^\sharp\big)\xrightarrow{\phi}\mathbb{Q}.$$

To simplify notation, from now on, we will denote $V\otimes H_+$ by S.

Theorem 3.3. *With the described brackets, $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L')$ is the L_∞-algebra determined by the condition $(\Lambda(\bar{S}^1\oplus S^{\geqslant 2}),\hat{d}_\psi)\cong\mathcal{C}^\infty(s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L'))$.*

Corollary 3.4. *Let $\gamma:\mathbb{L}(\mathcal{U})\to L'$ be a model for $f:X\to Y$, where $\mathbb{L}(\mathcal{U})$ is the minimal Lie model of X. Then $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L')$ is an L_∞-model of* $\operatorname{map}_f^*(X,Y)$ *with the L_∞ structure given by Definition* 3.1.

Proof. As shown at the end of Section 2, $(\Lambda(\bar{S}^1\oplus S^{\geqslant 2}),\hat{d}_\psi)$ is a Sullivan model for $\operatorname{map}_f^*(X,Y)$. Thus, Theorem 3.3 exhibits $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L')$ as an L_∞-model for $\operatorname{map}_f^*(X,Y)$. □

We have divided the proof of Theorem 3.3 into a sequence of technical lemmas.

Lemma 3.5. *Let $v\in V=(sL')^\sharp$ and $h\in H_+$. Then $\psi(v\otimes h)=\langle v;s\gamma(s^{-1}h)\rangle$.*

Proof. We have that

$$\psi(v\otimes h)=\phi(v\otimes h-x_{vh})=v\big(s\gamma q\big(s^{-1}h\big)\big)-\phi(x_{vh})=\big\langle v;s\gamma\big(s^{-1}h\big)\big\rangle-\phi(x_{vh}).$$

It remains to show that $\phi(x_{vh})=0$. Since $x_{vh}\in\Lambda W\otimes\Lambda^+(U\oplus\tilde{d}U)$; see Lemma 2.6, it is enough to check that $\phi(U\oplus\tilde{d}U)=0$. Let $v\otimes a$ be a generator of U. Then, $\phi(v\otimes a)=v(s\gamma q(s^{-1}a))=0$. Let $\tilde{d}(v\otimes a)$ be a generator of $\tilde{d}U$. Then

$$\phi\big(\tilde{d}(v\otimes a)\big)=\sum_{i,j}(-1)^{|w_i||z'_j|}\phi\big(u_i\otimes z'_j\big)\phi\big(w_i\otimes z''_j\big)+(-1)^{|v|}\phi(v\otimes\delta a),$$

where $d_2v=\sum_i u_iw_i$, and $\overline{\Delta}a=\sum_j z'_j\otimes z''_j$. The second term of the above expression can be written as

$$\begin{aligned}\phi(v\otimes\delta a)&=v\big(s\gamma q\big(s^{-1}\delta a\big)\big)=v\left(s\gamma q\left(-\partial s^{-1}a+\frac{1}{2}\sum_j(-1)^{|z'_j|}\big[s^{-1}z'_j,s^{-1}z''_j\big]\right)\right)\\&=\frac{1}{2}\sum_j(-1)^{|z'_j|}\big\langle v;s\big[\gamma q\big(s^{-1}z'_j\big),\gamma q\big(s^{-1}z''_j\big)\big]\big\rangle\\&=\sum_{i,j}(-1)^{|w_i||z'_j|+|z'_j|+|z''_j|}\phi\big(u_i\otimes z'_j\big)\phi\big(w_i\otimes z''_j\big).\end{aligned}$$

Then, since $|v|=|z'_j|+|z''_j|-1$, we have also that $\phi(\tilde{d}(v\otimes a))=0$. □

Consider the composition $\eta\wp:\Lambda(V\otimes B_+^\sharp)\to\Lambda S\to\Lambda(\bar{S}^1\oplus S^{\geqslant 2})$. Then for $v\otimes z\in\Lambda(V\otimes B_+^\sharp)$ we can write $\eta\wp(v\otimes z)=\sum_{j\geqslant 0}(\eta\wp(v\otimes z))_j$, where $(\eta\wp(v\otimes z))_j\in\Lambda^j(\bar{S}^1\oplus S^{\geqslant 2})$ has word length j. For clarity in the proofs of the next lemmas we introduce the following:

Definition 3.6. For any element $z\in B_+^\sharp$ we define recursively the number $\operatorname{depth}(z)$ in the following way:

(1) $\operatorname{depth}(z)=0$ if $z\in H_+\oplus A$ or $z=\delta a$, where $a\in A$ is a primitive element,
(2) $\operatorname{depth}(z)=1+\max\{\operatorname{depth}(z'_j),\operatorname{depth}(z''_j)\}$, if $z=\delta a$, where $a\in A$ is not primitive and $\overline{\Delta}a=\sum_j z'_j\otimes z''_j$.

Then, for example, an element $z=\delta a$ with $a\in A$ and $\overline{\Delta}a=\sum_j h'_j\otimes h''_j$, where $h'_j,h''_j\in H_+$, has $\operatorname{depth}(z)=1$.

Recall that given $z\in B_+^\sharp$, we will use the symbol $\lambda(z)$ to denote $\varphi^{-1}(\overline{s^{-1}z})$; see Section 2.1.

Lemma 3.7. *For any element $v\otimes z\in V\otimes B_+^\sharp$, we have $(\eta\wp(v\otimes z))_0=\langle v;s\gamma\lambda(z)\rangle$.*

Proof. We proceed by induction on the depth of z. Let $z \in B_+^\sharp$ with $\operatorname{depth}(z) = 0$. If $z = h$ with $h \in H_+$, then

$$\big(\eta\wp(v \otimes z)\big)_0 = \big(\eta(v \otimes h)\big)_0 = \psi(v \otimes h) = \big\langle v; s\gamma\big(s^{-1}h\big)\big\rangle = \big\langle v; s\gamma\lambda(z)\big\rangle.$$

If $z = a$ with $a \in A$ or $z = \delta a$, where $a \in A$ is a primitive element, then we have that $(\eta\wp(v \otimes z))_0 = 0 = \langle v; s\gamma\lambda(z)\rangle$, since $s^{-1}z \in I$ and in this case $\lambda(z) = 0$.

Suppose the lemma holds for any $z' \in B_+^\sharp$ with $\operatorname{depth}(z') < n$ and let $z \in B_+^\sharp$ with $\operatorname{depth}(z) = n$. Hence $z = \delta a$ with $\overline{\Delta} a = \sum_j z'_j \otimes z''_j$, and let $d_2 v = \sum_i u_i w_i$. Then

$$\begin{aligned}\big(\eta\wp(v \otimes z)\big)_0 &= (-1)^{|v|+1} \sum_{i,j} (-1)^{|w_i||z'_j|} \big(\eta\wp\big(u_i \otimes z'_j\big)\big)_0 \big(\eta\wp\big(w_i \otimes z''_j\big)\big)_0 \\ &= (-1)^{|v|+1} \sum_{i,j} (-1)^{|w_i||z'_j|} \big\langle u_i; s\gamma\lambda\big(z'_j\big)\big\rangle\big\langle w_i; s\gamma\lambda\big(z''_j\big)\big\rangle \\ &= (-1)^{|v|+1} \frac{1}{2} \sum_{i,j} \big\langle u_i w_i; s\gamma\lambda\big(z'_j\big) \wedge s\gamma\lambda\big(z''_j\big)\big\rangle \\ &= (-1)^{|v|+1+|z''_j|} \frac{1}{2} \sum_j \big\langle v; s\gamma\big[\lambda\big(z'_j\big), \lambda\big(z''_j\big)\big]\big\rangle = \big\langle v; s\gamma\lambda(z)\big\rangle,\end{aligned}$$

where we have used the inductive hypothesis in the second equality, the cocommutativity of $\overline{\Delta}$ in the third equality and Eq. (2.1) in the last equality. $\square$

Consider the isomorphism of graded vector spaces

$$\Phi : Der_\gamma\big(\mathbb{L}\big(s^{-1}H_+\big), L'\big) \xrightarrow{\cong} S^\sharp \tag{3.2}$$

given by $\Phi(\theta)(v \otimes h) = (-1)^{|\theta|(|v|+1)} v(s\theta(s^{-1}h))$, and the corresponding pairing $\langle\,;\,\rangle : S \times Der_\gamma(\mathbb{L}(s^{-1}H_+), L') \to \mathbb{Q}$. Extend this to $(k+1)$-linear maps

$$\Lambda^k S \times Der_\gamma\big(\mathbb{L}\big(s^{-1}H_+\big), L'\big) \times \cdots \times Der_\gamma\big(\mathbb{L}\big(s^{-1}H_+\big), L'\big) \to \mathbb{Q},$$

$$\langle v_1 \otimes h_1 \wedge \cdots \wedge v_k \otimes h_k; \theta_1, \ldots, \theta_k\rangle = \sum_{\sigma \in S_k} \epsilon_\sigma \langle v_{\sigma(1)} \otimes h_{\sigma(1)}; \theta_k\rangle \cdots \langle v_{\sigma(k)} \otimes h_{\sigma(k)}; \theta_1\rangle.$$

Lemma 3.8. *For any element $v \otimes z \in V \otimes B_+^\sharp$ and derivations $\theta_1, \ldots, \theta_k \in Der_\gamma(\mathbb{L}(s^{-1}H_+), L')$, we have*:

$$\big\langle\big(\eta\wp(v \otimes z)\big)_k; \theta_1, \ldots, \theta_k\big\rangle = (-1)^\epsilon \big\langle v; s\{\theta_1, \ldots, \theta_k\}\lambda(z)\big\rangle,$$

where $\epsilon = (|\theta_1| + \cdots + |\theta_k|)(|z| + 1)$.

Proof. We proceed by induction on k. If $k = 1$, then we prove this case by induction on $\operatorname{depth}(z)$. Let $z \in B_+^\sharp$ with $\operatorname{depth}(z) = 0$. If $z = h$ with $h \in H_+$, we have

$$\begin{aligned}\big\langle\big(\eta\wp(v \otimes z)\big)_1; \theta_1\big\rangle &= (-1)^{|\theta|(|z|+1)} v\big(s\theta_1\big(s^{-1}h\big)\big) = (-1)^{|\theta|(|z|+1)} \big\langle v; s\theta_1\big(s^{-1}h\big)\big\rangle \\ &= (-1)^{|\theta|(|z|+1)} \big\langle v; s\{\theta_1\}\lambda(z)\big\rangle.\end{aligned}$$

If $z \in A$ or $z \in \delta A$, where $z = \delta a$ with a primitive, then $\langle(\eta\wp(v \otimes z))_1; \theta_1\rangle = 0 = \langle v; s\{\theta_1\}\lambda(z)\rangle$, directly from the definitions of the operator $\wp$ and the isomorphism φ^{-1}.

Now assume the lemma holds for $z' \in B_+^\sharp$ with $\operatorname{depth}(z') < n$ and let $z \in B_+^\sharp$ with $\operatorname{depth}(z) = n$. Let $z = \delta a$, where $\overline{\Delta} a = \sum_j z'_j \otimes z''_j$, and let $d_2 v = \sum_i u_i w_i$.

Here, for simplicity in the notation we shall omit signs and just write $\pm$. However, a careful use of the Koszul convention leads to proper sign adjustments. Then

$$\begin{aligned}\big\langle\big(\eta\wp(v \otimes z)\big)_1; \theta_1\big\rangle = {}&\pm \sum_{|u_i \otimes z'_j| = 0} \big(\eta\wp\big(u_i \otimes z'_j\big)\big)_0 \big\langle\big(\eta\wp\big(w_i \otimes z''_j\big)\big)_1; \theta_1\big\rangle \\ &\pm \sum_{|w_i \otimes z''_j| = 0} \big\langle\big(\eta\wp\big(u_i \otimes z'_j\big)\big)_1; \theta_1\big\rangle \big(\eta\wp\big(w_i \otimes z''_j\big)\big)_0\end{aligned}$$

$$= \pm \sum_{|u_i \otimes z'_j|=0} \langle u_i; s\gamma\lambda(z'_j)\rangle\langle w_i; s\{\theta_1\}\lambda(z''_j)\rangle \pm \sum_{|w_i \otimes z''_j|=0} \langle u_i; s\{\theta_1\}\lambda(z'_j)\rangle\langle w_i; s\gamma\lambda(z''_j)\rangle$$
$$= \pm \sum_{i,j} \frac{1}{2}\langle v; s[\theta_1\lambda(z'_j), \gamma\lambda(z''_j)]\rangle \pm \sum_{i,j} \frac{1}{2}\langle v; s[\gamma\lambda(z'_j), \theta_1\lambda(z''_j)]\rangle$$
$$= \pm\langle v; s\{\theta_1\}\lambda(z)\rangle.$$

The lemma now follows from the inductive hypothesis by a similar computation. □

Lemma 3.9. *Let $v \otimes h \in V \otimes H_+ = (sL')^\sharp \otimes H_+$ and let $\theta \in \mathcal{D}er_\gamma(\mathbb{L}(s^{-1}H_+), L')$. Then $\langle (\hat{d}_\psi)_1(v \otimes h); \theta\rangle = (-1)^{|v|+|h|}\langle v \otimes h; \delta\theta\rangle$.*

Proof. By Lemma 2.8 and the comments made at the end of Section 2:

$$(\hat{d}_\psi)_1(v \otimes h) = d_1 v \otimes h + \left(\eta \sum_{i,j} (-1)^{|w_i||z'_j|} \wp(u_i \otimes z'_j)\wp(w_i \otimes z''_j)\right)_1.$$

Then, on one hand:

$$\langle d_1 v \otimes h; \theta\rangle = (-1)^{|\theta|(|h|+1)}\langle d_1 v; s\theta(s^{-1}h)\rangle$$
$$= (-1)^{|\theta|(|h|+1)+|v|+1}\langle v; s\partial\theta(s^{-1}h)\rangle = (-1)^{|v|+|h|}\langle v \otimes h; \partial\theta\rangle.$$

On the other hand, by Lemmas 3.7 and 3.8:

$$\left\langle \left(\eta \sum_{i,j} (-1)^{|w_i||z'_j|} \wp(u_i \otimes z'_j)\wp(w_i \otimes z''_j)\right)_1; \theta \right\rangle$$
$$= \frac{1}{2}\left(\sum_{i,j} (-1)^{|\theta|(|z''_j|+1)}\langle u_i w_i; s\gamma\lambda(z'_j) \wedge s\theta\lambda(z''_j)\rangle + \sum_{i,j} (-1)^{|\theta|(|z'_j|+|z''_j|+1)}\langle u_i w_i; s\theta\lambda(z'_j) \wedge s\gamma\lambda(z''_j)\rangle\right)$$
$$= (-1)^{|\theta||h|+|\theta|+|h|}\left\langle v; s\theta\left(\frac{1}{2}\sum_j (-1)^{|z'_j|}[\lambda(z'_j), \lambda(z''_j)]\right)\right\rangle$$
$$= (-1)^{|\theta||h|+|\theta|+|h|}\langle v; s\theta(\partial s^{-1}h)\rangle = -\langle v \otimes h; \theta\partial\rangle.$$

The claim follows by putting both expressions together. □

Lemma 3.10. *Let $v \otimes h \in (sL)^\sharp \otimes H_+$, and let $\theta_1, \ldots, \theta_k \in \mathcal{D}er_\gamma(\mathbb{L}(s^{-1}H_+), L')$. Then, for $k > 1$ we have that*

$$\langle (\hat{d}_\psi)_k(v \otimes h); \theta_1, \ldots, \theta_k\rangle = (-1)^\epsilon\langle v \otimes h; s\ell_k(s^{-1}\theta_1, \ldots, s^{-1}\theta_k)\rangle,$$

where ϵ is the sign provided by the Koszul convention and described in Proposition 2.3.

Proof. Again by Lemma 2.8 and the comments made at the end of Section 2:

$$(\hat{d}_\psi)_k(v \otimes h) = \left(\eta \sum_{i,j} (-1)^{|w_i||z'_j|} \wp(u_i \otimes z'_j)\wp(w_i \otimes z''_j)\right)_k.$$

The rest is a straightforward computation using Lemmas 3.7 and 3.8 as in the previous lemma. □

Proof of Theorem 3.3. The isomorphism (3.2) restricts to an isomorphism

$$\Phi : ZDer_\gamma(\mathbb{L}(s^{-1}H_+), L')_1 \oplus Der_\gamma(\mathbb{L}(s^{-1}H_+), L')_{\geqslant 2} \xrightarrow{\cong} (\bar{S}^1 \oplus S^{\geqslant 2})^\sharp,$$

since due to Lemma 3.9

$$\Phi(\theta)(D(v \otimes h)) = \Phi(\theta)((\hat{d}_\psi)_1(v \otimes h)) = \pm\langle (\hat{d}_\psi)_1(v \otimes h); \theta\rangle = \pm\langle v \otimes h; \delta\theta\rangle.$$

This expression vanishes for every $v \otimes h \in S^0$ if and only if $\delta\theta = 0$, that is, $\theta \in ZDer_\gamma(\mathbb{L}(s^{-1}H_+), L')_1$.

Then we have the isomorphism of commutative graded algebras

$$\mathcal{C}^\infty(s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(s^{-1}H_+), L')) \cong \Lambda(\bar{S}^1 \oplus S^{\geqslant 2}).$$

It remains to be shown that the differential $\hat{d}_\psi$ coincides with the differential on $\mathcal{C}^\infty(s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(s^{-1}H_+), L'))$. However, by Proposition 2.3, this is precisely Lemmas 3.9 and 3.10. □

4. Whitehead-length and H-space structures on mapping spaces

In this section $(\mathbb{L}(\mathcal{U}),\partial)$, $(L',0)$ and $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L')$ will denote the minimal Quillen model of X, a DGL model of Y, and the L_∞-model of $\mathrm{map}^*_f(X,Y)$ given by Theorem 3.3, respectively.

We prove now Theorem 1.4. The first assertion of this theorem is proved by means of the isomorphism of Lie algebras

$$H_*\big(s^{-1}\mathcal{D}er_\gamma\big(\mathbb{L}(\mathcal{U}),L'\big)\big)\cong\pi_*\big(\Omega\,\mathrm{map}^*_f(X,Y)_{\mathbb{Q}}\big),$$

which is a direct consequence of Theorem 3.3. Therefore we shall use only the bracket ℓ_2 of Definition 3.1 to prove this part. However, to prove the second assertion we shall use all the information provided by ℓ_k for $k\geqslant 2$.

Proof of Theorem 1.4. If X is contractible, then $\mathrm{map}^*_f(X,Y)$ is contractible and all Whitehead products in $\mathrm{map}^*_f(X,Y)$ vanish.

If X has a Quillen minimal model of the form $(\mathbb{L}(\mathcal{U}),0)$, all Whitehead products vanish directly from definition of the bracket in $s^{-1}\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L')$. The inequality

$$\mathrm{Wh}_{\mathbb{Q}}\big(\mathrm{map}^*_f(X,Y)\big)\leqslant\frac{\mathrm{Wh}_{\mathbb{Q}}(Y)-1}{\infty-1}+1=1$$

gives the same answer in both cases.

Otherwise, suppose $\mathrm{Wh}_{\mathbb{Q}}(Y)=p$, $\mathrm{bl}_{\mathbb{Q}}(X)=q\geqslant 2$ and consider m cycles

$$s^{-1}\theta_1,\dots,s^{-1}\theta_m\in s^{-1}\mathcal{D}er_\gamma\big(\mathbb{L}(\mathcal{U}),L'\big),$$

where m satisfies the relation $(m-1)(q-1)+1>p$. Given $u\in\mathcal{U}$, we apply recursively Definition 3.1 to

$$s\ell_2\big(s^{-1}\theta_m,\dots,\ell_2\big(s^{-1}\theta_2,s^{-1}\theta_1\big)\dots\big)(u).$$

Any term of the resulting formula is zero or an iterated Whitehead bracket in L' of length $>p$. Then, any iterated Whitehead bracket of $m>\frac{p-1}{q-1}+1$ cycles is zero. This proves the first assertion.

Let $\theta_1,\dots,\theta_k\in\mathcal{D}er_\gamma(\mathbb{L}(\mathcal{U}),L')$ and $h\in H_+$. By Definition 3.1, any term of

$$s\ell_k\big(s^{-1}\theta_1,\dots,s^{-1}\theta_k\big)\big(s^{-1}h\big)=\pm\{\theta_1,\dots,\theta_k\}\big(\partial s^{-1}h\big)$$

for $k\geqslant 2$ is an iterated Whitehead bracket in L' of length $\mathrm{bl}_{\mathbb{Q}}(X)>\mathrm{Wh}_{\mathbb{Q}}(Y)$, and thus is zero. Then, for any $v\in(sL')^\sharp$, by Lemma 3.10

$$\begin{aligned}\big\langle(\hat d_\psi)_k(v\otimes h);\theta_1,\dots,\theta_k\big\rangle&=\pm\big\langle v\otimes h;s\ell_k\big(s^{-1}\theta_1,\dots,s^{-1}\theta_k\big)\big\rangle\\&=\pm v\big(s\ell_k\big(s^{-1}\theta_1,\dots,s^{-1}\theta_k\big)\big(s^{-1}h\big)\big)=0.\end{aligned}$$

This implies that $(\hat d_\psi)_k$ vanishes for all $k\geqslant 2$. Then, the differential on the minimal model of $\mathrm{map}^*_f(X,Y)$ vanishes and the theorem follows (see [10, §12, Example 3]). $\square$

Remark. The first assertion of Theorem 1.4 improves a theorem of Lupton and Smith [18] proving the inequality $\mathrm{Wh}_{\mathbb{Q}}(\mathrm{map}^*_f(X,Y))\leqslant\mathrm{Wh}_{\mathbb{Q}}(Y)$ under the same assumptions.

If $\mathrm{bl}_{\mathbb{Q}}(X)>\mathrm{Wh}_{\mathbb{Q}}(Y)$, from the first assertion of Theorem 1.4 we have that all Whitehead products in $\pi_{\geqslant 2}(\mathrm{map}^*_f(X,Y))\otimes\mathbb{Q}$ vanish. However, the second assertion is stronger and can be seen as a dual result of [5, Theorem 3].

Finally, we show spaces for which the difference between the upper bound given by Theorem 1.4 and the one given in [18] can be any positive integer.

Example 4.1. Consider the following sequence of sphere bundles

$$S^{2n+3}\to Y_{n+1}\to Y_n,\quad n\geqslant 1,$$

where $Y_1=(\Lambda u_0,u_1,d)$, $|u_0|=|u_1|=3$, $d=0$, and $Y_n=(\Lambda u_0,u_1,\dots,u_n,d)$, $|u_n|=2n+1$, $du_n=u_0u_{n-1}$ (we use the same letter Y_n, to denote the space and its model).

Recall that the Whitehead product in $\pi_*(Y_n)$ is dual to the quadratic part of the differential of its minimal Sullivan model $(\Lambda U,d)$ [10, Proposition 13.16]. Then it is clear that $\mathrm{Wh}_{\mathbb{Q}}(Y_n)=n$.

Let $n\geqslant 3$ and define the space $X_n=S^3_a\vee S^3_b\cup_{[\alpha,\dots[\alpha,\beta]_W]_W}D^{2n+2}$, where $\alpha,\beta\in\pi_3(S^3_a\vee S^3_b)$ are represented by S^3_a and S^3_b respectively, and the iterated bracket of n factors $[\alpha,\dots[\alpha,\beta]_W]_W\in\pi_{2n+1}(S^3_a\vee S^3_b)$. Then X_n has a minimal Lie model of the form $(\mathbb{L}(a,b,c_n),\partial)$ with $|a|=|b|=2$, $|c_n|=2n+1$ and $\partial c_n=[a,\dots[a,b]]$. It is clear that $\mathrm{bl}_{\mathbb{Q}}(X_n)=n$.

Then, for any $f:X_n\to Y_n$, by Theorem 1.4, we have a constant upper bound $\mathrm{Wh}_{\mathbb{Q}}(\mathrm{map}^*_f(X_n,Y_n))\leqslant 2$. However, the upper bound given by the inequality $\mathrm{Wh}_{\mathbb{Q}}(\mathrm{map}^*_f(X_n,Y_n))\leqslant\mathrm{Wh}_{\mathbb{Q}}(Y_n)$, is the positive integer n.

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